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THE INFLUENCE OF SHEAR DEFORMATIONS ON THE STABILITY OF THIN-WALLED BEAMS UNDER NON-CONSERVATIVE LOADING

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Abstract—This paper analyses the influence of shear deformations on the stability of thin-walled beams of open section subjected to follower-type loads. The warping of the cross-sections is assumed to depend on the shear forces as well as on both primary and secondary torsion. The internal stresses are consistently deduced from the kinematic field. An algorithm is presented which works in the framework of the least squares method making use of a vector iteration technique. An extensive comparison is performed with the results offered by the classical models.

1. INTRODUCTION

Starting from the pioneering analysis by Nikolai (1928) of a cantilever beam subjected to a torque vector acting at the free end, subsequent works of Russian and German schools (Pflüger, 1950; Beck, 1952) were devoted to single problems of non-conservative mechanical systems. A comprehensive clarification of the basic concepts of elastic stability came from the work of Ziegler (1953, 1956), who offered a classification both of the mechanical systems and of the related methods of analysis. It was, then, more evident that in the presence of non-conservative forces, the system may lack adjacent configurations of static equilibrium and the instability may consist of a vibrational motion of increasing amplitude (flutter).

The problem of bending-torsional flutter of an elastic thin-walled beam as well as that of a linearly elastic continuum subjected to follower type loads was first formulated by Bolotin (1963), whose monograph gave a new impulse to these studies (Leipholz, 1964; Herrmann and Jong, 1965; Como, 1966; Augusti, 1966; Nemat-Nasser and Herrmann, 1966a). In this context, the equations of motion with the corresponding boundary conditions define a non-self-adjoint boundary value problem since circulatory forces cannot, in general, be associated with a stationary single-valued functional dependent on generalized displacements only. Hence classical variational principles do not hold true and numerical solutions were often obtained by making use of the Galerkin–Petrov method or of the incremental virtual work principle when a convenient foundation for an F.E. analysis was required (Argyris *et al.*, 1981; Argyris and Symeonidis, 1981). Further F.E. formulations were given by Barsoum (1971) and more recently by Attard and Somervaille (1987). Convergence investigations of the Galerkin method can be found in Leipholz (1962, 1963, 1983) and Levinson (1966).

An alternative procedure has been followed by several authors (Nemat-Nasser and Herrmann, 1966b; Prasad and Herrmann, 1969, 1972; Dubey and Leipholz, 1975) in the attempt to extend the variational formulations to non-conservative problems in the context of the linearized theory of elastic stability. All these formulations, which make use of adjoint variational methods, can be seen as particular cases of a more general approach given by Telega (1979). An iterative method for the solution of non-conservative stability problems was recently proposed by Xiong *et al.* (1989) by solving a set of related conservative problems. In the same framework a least squares quadratic functional (Altman and Marmo de Oliveira, 1984), rather than the adjoint equation, was employed. An original approach was proposed by Leipholz (1980), who introduced a suitable function space supplied with an appropriate "semi-scalar" product, where non-conservative systems behave like conservative ones.

Sufficient conditions for the stability of non-conservative systems were given in Benvenuto and Corsanego (1974) by making use of the second Liapunov method.

Sufficient conditions for self-adjointness of linear operators, in the presence of live loads, in a class of small incremental displacements, were determined by Capriz and Podio Guidugli (1981). Recently, Tonti (1984) formulated a general procedure in order to obtain variational formulations for almost non-linear, non-self-adjoint systems by adopting abstract spaces involving variations of operators rather than functionals. Starting from Tonti's formulation, Alliney and Tralli (1984, 1986) deduced suitable F.E. models and extended the quoted procedure obtaining eigenvalue problems with symmetric but not definite matrices.

A different approach consisting of a direct *a posteriori* symmetrization of the load correction matrix was proposed by Mang and Gallagher (1983). Nonetheless, it must be remarked that such a procedure is of cumbersome application (Klee and Wriggers, 1983) and physically meaningful only for divergence-type systems (Argyris *et al.*, 1981).

Recent developments in the field of space mechanics, hydro and aeroelasticity, mechanical and structural engineering have underlined the increasing importance of the structures subjected to non-conservative forces and the need for proper methods of analysis. Cable loads, pressure loads acting on offshore or submarine structures, on suspended bridges or on cooling towers are typical examples of non-conservative loads considered by structural engineering.

In the more specific field of stability analysis of thin-walled beams, it must be remembered that the beam model has been refined by Nemat-Nasser (1967) and Kounadis (1977) by including the effects of shear deformations according to Timoshenko theory or by Nemat-Nasser and Herrmann (1966a) and Nemat-Nasser and Tsai (1969) by including the influence of warping rigidity. More recently, Kounadis and Sophianopoulos (1986) analyzed the effect of axial inertia on the bending eigenfrequencies of a two-bar frame according to the Timoshenko beam model (for conservative loads only).

In the present paper, all these aspects are unified by the authors in order to analyze the dynamic response of an open symmetric thin-walled beam subjected to any given distribution of follower loads acting in a symmetry plane, by making use of a description of the axial displacement field recently proposed by Laudiero and Savoia (1990a,b). According to this model, for a beam subjected to non-uniform bending and torsion, the local shear deformations of the middle surface of the beam are taken into account; hence, with reference to torsion, a "secondary" warping is considered which is superimposed to the "primary" warping given by the theory of sectorial areas. The warping functions are assumed to depend on the local beam resultants only, in the context of the de St. Venant model. These functions yield an approximate representation of the infinite warping functions obtained by Capurso (1964) through an eigenfunction series expansion. In the present paper, the authors make use of Hamilton's law (Bailey, 1975) to obtain a variational equation from which 10 differential equations of motion are deduced. Partial decoupling is obtained since the beam is assumed to exhibit one plane of symmetry. The numerical solution is then pursued through the classical trigonometric series expansion resulting in a system of linear homogeneous equations called "generalized and extended equations of Ritz" by Leipholz (1983). The corresponding eigenvalue problem is ruled by a non-symmetric and strongly ill-conditioned matrix. All the boundary (linear and homogeneous) conditions which are not satisfied *a priori* by the coordinate functions are enforced through a projection technique onto the subspace defined by the mentioned constraints. The solution of an eigenvalue problem ruled by a full and non-symmetric matrix is a hard task, especially for an ill-conditioned problem. For this purpose, an algorithm has been developed (Alliney *et al.*, 1990) which makes use, in the framework of the least squares method, of a vector iteration technique.

A variety of example problems illustrate the response of a cantilever beam, having various cross-sections subjected to different load conditions. The influence of shear deformations on (possibly coupled) flexural and torsional modes is investigated and extensive comparisons are made with the classical models.

2. KINEMATICS

Figure 1 shows a typical open thin-walled prismatic beam. A right-handed orthogonal coordinate system C_0xyz is adopted, where x and y are the centroidal principal axes. Positive displacement components are illustrated in the same figure. The coordinates of the shear centre $S(x_S, y_S)$, in the local reference system Pnp, shown in Fig. 2, are given by:

$$r = (x - x_S)\frac{dy}{ds} - (y - y_S)\frac{dx}{ds}$$

$$h = (x - x_S)\frac{dx}{ds} + (y - y_S)\frac{dy}{ds}, \qquad (1)$$

where s is a curvilinear coordinate lying on the middle line and the axes p and n are tangent and normal to the middle line respectively.

It is assumed that the cross-sections are rigid in their own planes whereas the shearing deformation on the middle surface is not constrained to zero (assumption a). Moreover, the displacement gradient components corresponding to the loaded configuration are assumed to be negligible with respect to unity (assumption b).

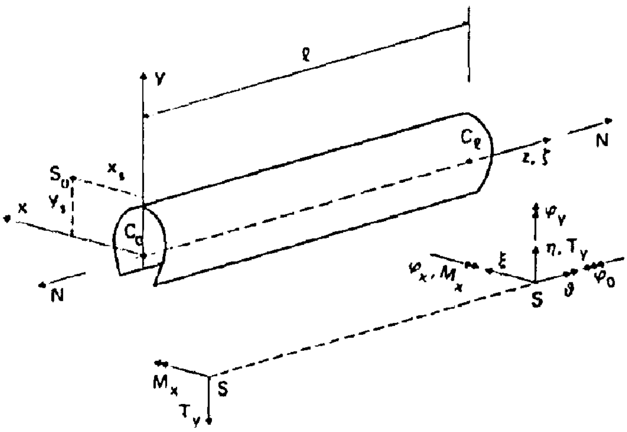


Fig. 1. Beam geometry and positive sense of displacements.

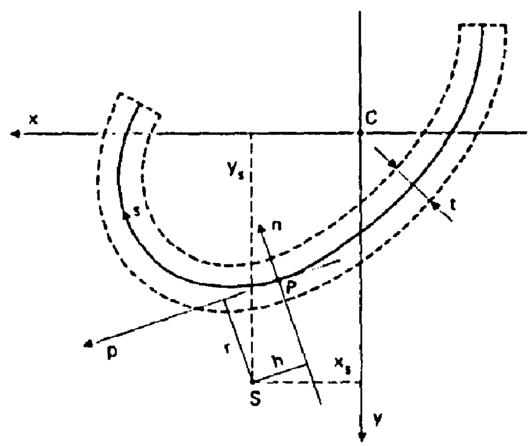


Fig. 2. Typical cross-section and local reference system.

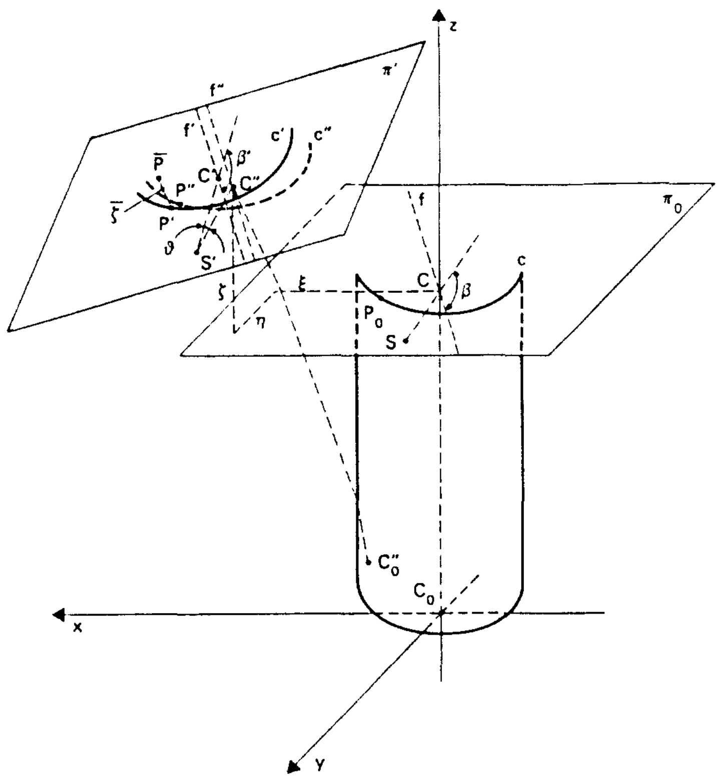


Fig. 3. Displacement field of the beam.

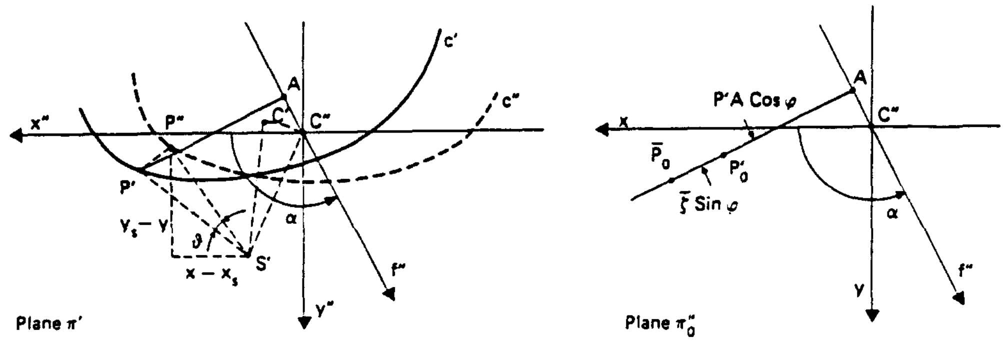
If we consider assumption a, let π' be the plane which defines the average displacement of the cross-section contour c (Fig. 3). C' and S' denote the location of the centroid and the shear centre of c' on the plane π'. By f' we will denote the intersection between π' and the plane containing C' parallel to the original plane π_0 of the cross-section, whereas f will denote the straight line parallel to f' and containing the original centroid, C. The straight line f is oriented so that the x axis rotates counterclockwise by the angle α, less than π, to coincide with f. Let us define the angle θ like $\theta = \beta' - \beta$, where β' and β are shown in Fig. 3, and rotate c' by an angle $-\theta$ around S' where the positive rotation is assumed to be counterclockwise. As a consequence, C' and c' move to C'' and c'' respectively and the displacement functions ξ, η and ζ are defined as the components of CC'' along x, y and z respectively. By f'' we will denote the intersection between π' and the horizontal plane π'_0 containing C''.

Let P be a typical point belonging to c, $\bar{P}$ its final position and P' the projection of $\bar{P}$ on the plane π'. Because of the rotation $-\theta$ around S', the point P' arrives at P'' which, with respect to the swung axes x' and y', keeps the same coordinates x and y of the point P in the unstrained configuration (Fig. 4a). Analogously, S' keeps the same coordinates as S. Hence, the coordinates of P', with respect to x' and y', take the form:

$$\begin{aligned} x' &= x-(y-y_S)\sin\theta-(x-x_S)(1-\cos\theta) \\ y' &= y+(x-x_S)\sin\theta-(y-y_S)(1-\cos\theta). \end{aligned} \tag{2}$$

With reference to Fig. 4a, we have:

$$\begin{aligned} P'A &= -x'\sin\alpha+y'\cos\alpha \\ C''A &= -x'\cos\alpha-y'\sin\alpha. \end{aligned} \tag{3}$$



(a) (b)

Fig. 4. In-plane section displacements.

Let φ be the angle by which the plane π''_0 rotates around f'' to coincide with π'. Moreover, by $\bar{\zeta}$ we will denote the out-of-plane displacement $P'\bar{P}$. If P'_0 and $\bar{P}_0$ are the projections of P' and $\bar{P}$ on π''_0, we obtain (Fig. 4b):

$$\bar{P}_0 A = P'A \cos\varphi + \bar{\zeta} \sin\varphi$$
$$\bar{P}_0 \bar{P} = -P'A \sin\varphi + \bar{\zeta} \cos\varphi. \tag{4}$$

Finally, considering that the coordinates of $\bar{P}$ in the reference $C''xyz$ are:

$$x_0 = -\bar{P}_0 A \sin\alpha - C''A \cos\alpha$$
$$y_0 = \bar{P}_0 A \cos\alpha - C''A \sin\alpha$$
$$z_0 = \bar{P}_0 \bar{P}, \tag{5}$$

and keeping in mind that the coordinates of C'' in the reference C_0xyz are given by ξ, η, $z+\zeta$, the displacement components of P along the axes x, y and z are given by:

$$u_x = \xi - x + x'(\cos\varphi \sin^2\alpha + \cos^2\alpha) + y'(1-\cos\varphi)\sin\alpha\cos\alpha - \bar{\zeta}\sin\varphi\sin\alpha$$
$$u_y = \eta - y + y'(\cos\varphi\cos^2\alpha + \sin^2\alpha) + x'(1-\cos\varphi)\sin\alpha\cos\alpha + \bar{\zeta}\sin\varphi\cos\alpha$$
$$u_z = \zeta + x'\sin\varphi\sin\alpha - y'\sin\varphi\cos\alpha + \bar{\zeta}\cos\varphi, \tag{6}$$

where x' and y' are given by eqns (2).

If we consider assumption b, for an arbitrary variation of $\xi(z)$, $\eta(z)$, $\zeta(z)$, $\varphi(z)$, $\theta(z)$ and $\bar{\zeta}(z,s)$ in a neighborhood of the static equilibrium configuration and for any $\alpha(z)$, the linear and quadratic terms of the displacement components are:

$$u_x^{(1)} = \xi - (y - y_S)\theta$$
$$u_y^{(1)} = \eta + (x - x_S)\theta$$
$$u_z^{(1)} = \zeta - x\varphi_y - y\varphi_x + \bar{\zeta}, \tag{7'}$$

and

$$u_x^{(2)} = -\tfrac{1}{2}x\varphi_y^2 - \tfrac{1}{2}y\varphi_x\varphi_y - \tfrac{1}{2}(x-x_S)\theta^2 + \bar{\zeta}\varphi_y$$
$$u_y^{(2)} = -\tfrac{1}{2}y\varphi_x^2 - \tfrac{1}{2}x\varphi_x\varphi_y - \tfrac{1}{2}(y-y_S)\theta^2 + \bar{\zeta}\varphi_x$$
$$u_z^{(2)} = -(x-x_S)\varphi_x\theta + (y-y_S)\varphi_y\theta, \tag{7''}$$

where

$$\varphi_x = \varphi \cos \alpha, \quad \varphi_y = -\varphi \sin \alpha \tag{8}$$

are the first order rotations of the cross-section around the axes x and y respectively. The definition (8) implies that the corresponding positive rotation vectors are oriented like negative x and positive y axes respectively. Finally, the linear part of the out-of-plane displacement $\bar{\zeta}$ is cast in the form (Laudiero and Savoia, 1990a):

$$\bar{\zeta} = -\varphi_\omega(z)\omega(s) + \chi_x(z)\psi_x(s) + \chi_y(z)\psi_y(s) + \chi_\omega(z)\psi_\omega(s), \tag{9}$$

where $\omega(s)$ is assumed to coincide with the sectorial areas. The functions φ_x and φ_y are able to reproduce the Reissner–Mindlin formulation (Mindlin, 1951); when they are made coincident with η' and ξ' respectively,† the Euler–Bernoulli model is obtained. Analogously, when the function φ_ω is made coincident with θ', the Vlasov formulation is recovered. The terms $\chi_x(z)\psi_x(s)$ and $\chi_y(z)\psi_y(s)$ take into account the shear strains due to the beam forces T_x and T_y, whereas the term $\chi_\omega(z)\psi_\omega(s)$ (secondary warping) plays the analogous role with respect to the shear stresses arising from non-uniform torsion. By making use of eqns (7′) and (9) the linearized displacement components of the cross-section contour in the local reference system $Pnpz$ are given by:

$$\begin{aligned}
u_n^{(1)} &= \xi \frac{dy}{ds} - \eta \frac{dx}{ds} + \theta h \\
u_p^{(1)} &= \xi \frac{dx}{ds} + \eta \frac{dy}{ds} - \theta r \\
u_z^{(1)} &= \zeta - \varphi_y x - \varphi_x y - \varphi_\omega \omega + \chi_x \psi_x + \chi_y \psi_y + \chi_\omega \psi_\omega .
\end{aligned} \tag{10}$$

The warping functions ψ_x, ψ_y, ψ_ω are obtained under the assumption that they depend only on the local beam resultants. Analogously with the de St. Venant model, assuming bending and twisting moments linearly varying along the beam axis and making use of eqns (10), the warping functions can be obtained by means of the following relations (Laudiero and Savoia, 1990b):

$$\psi_i(s) = \bar{\psi}_i(s) - \sum_j j(s)c_{ij} - c_i \quad (i, j = x, y, \omega) \tag{11}$$

where

$$\frac{d}{ds}\bar{\psi}_x = -\frac{D_x S_y}{J_y t}, \quad \frac{d}{ds}\bar{\psi}_y = -\frac{D_y S_x}{J_x t}, \quad \frac{d}{ds}\bar{\psi}_\omega = -\frac{D_\omega S_\omega}{J_\omega t}, \tag{12}$$

$t(s)$ is the thickness of the cross-section,

$$D_x = \int_A \left(\frac{dx}{ds}\right)^2 dA, \quad D_y = \int_A \left(\frac{dy}{ds}\right)^2 dA, \quad D_\omega = \int_A \left(\frac{d\omega}{ds}\right)^2 dA, \tag{13}$$

A is the area of the cross-section and S_x, S_y, S_ω and J_x, J_y, J_ω are the first and the second area and sectorial moments respectively.

The three constants c_i and the nine constants c_{ij} can be determined by imposing the orthogonality conditions:

† Superscript ( )′ means derivative with respect to z.

$$\int_A \psi_i \, \mathrm{d}A = 0$$
$$\int_A j\psi_i \, \mathrm{d}A = 0 \qquad (i, j = x, y, \omega), \tag{14}$$

resulting in:

$$c_i = \frac{1}{A}\int_A \bar{\psi}_i \, \mathrm{d}A$$
$$c_{ij} = \frac{\int_A j\bar{\psi}_i \, \mathrm{d}A}{\int_A j^2 \, \mathrm{d}A} \qquad (i, j = x, y, \omega). \tag{15}$$

3. STRAIN–DISPLACEMENT RELATIONS

Adopting a Lagrangian description, the linear and quadratic terms of the Green strain tensor are given by:

$$\boldsymbol{\varepsilon}^{(1)} = \tfrac{1}{2}(\nabla \mathbf{u}^{(1)} + \mathbf{u}^{(1)}\nabla)$$
$$\boldsymbol{\varepsilon}^{(2)} = \tfrac{1}{2}\nabla \mathbf{u}^{(1)} \cdot \mathbf{u}^{(1)}\nabla + \tfrac{1}{2}(\nabla \mathbf{u}^{(2)} + \mathbf{u}^{(2)}\nabla) \tag{16a,b}$$

where the operator ∇ means gradient (or its transpose when it is postponed). Making use of the linearized displacement field (10), eqn (16a) yields:

$$\varepsilon_{ss}^{(1)} = \varepsilon_{nn}^{(1)} = 0$$
$$\varepsilon_{zz}^{(1)} = \zeta' - \varphi_y' x - \varphi_x' y - \varphi_\omega' \omega + \chi_x' \psi_x + \chi_y' \psi_y + \chi_\omega' \psi_\omega$$
$$\varepsilon_{zs}^{(1)} = \frac{1}{2}\left[(\xi' - \varphi_y)\frac{\mathrm{d}x}{\mathrm{d}s} + (\eta' - \varphi_x)\frac{\mathrm{d}y}{\mathrm{d}s} + (\theta' - \varphi_\omega)\frac{\mathrm{d}\omega}{\mathrm{d}s} + \chi_x \frac{\mathrm{d}}{\mathrm{d}s}(\psi_x) + \chi_y \frac{\mathrm{d}}{\mathrm{d}s}(\psi_y) + \chi_\omega \frac{\mathrm{d}}{\mathrm{d}s}(\psi_\omega)\right]. \tag{17'}$$

Moreover, recalling that the cross-section does not distort and assuming that any straight line remains normal to the middle surface of the beam, we have:

$$\varepsilon_{sn}^{(1)} = \varepsilon_{zn}^{(1)} = 0. \tag{17''}$$

Analogously, assuming the linearized displacement field, eqn (16b) becomes:

$$\varepsilon_{zs}^{(2)} = \frac{1}{2}\left\{-\xi'\theta \frac{\mathrm{d}y}{\mathrm{d}s} + \eta'\theta \frac{\mathrm{d}x}{\mathrm{d}s} - \theta\theta' h + u_{z,z}^{(1)} u_{z,s}^{(1)}\right\}$$
$$\varepsilon_{zn}^{(2)} = \frac{1}{2}\left\{-\xi'\theta \frac{\mathrm{d}x}{\mathrm{d}s} - \eta'\theta \frac{\mathrm{d}y}{\mathrm{d}s} + \theta\theta' r + u_{z,z}^{(1)} u_{z,n}^{(1)}\right\}$$
$$\varepsilon_{zz}^{(2)} = \tfrac{1}{2}\{\xi'^2 + \eta'^2 - 2(y - y_S)\xi'\theta + 2(x - x_S)\eta'\theta + [(x - x_S)^2 + (y - y_S)^2]\theta'^2 + u_{z,z}^{(1)} u_{z,z}^{(1)}\}. \tag{18'}$$

Moreover, in accordance with the kinematic assumptions, the following is adopted:

$$\varepsilon_{ss}^{(2)} = \varepsilon_{nn}^{(2)} = \varepsilon_{ns}^{(2)} = 0. \tag{18''}$$

Since $u_{,z}^{(1)}$ is expected to be very small compared with unity, the terms of eqns (18′) containing such a derivative will be neglected in the following.

4. THE LOAD CONDITION

The loading condition of the beam will be defined in terms of a distribution of a line load $\mathbf{q}$ and in two distributions of loads acting at the end sections, $\mathbf{f}(0)$ and $\mathbf{f}(l)$, having y and z components. Both the load $\mathbf{q}$ and the loads $\mathbf{f}$ are assumed to act in a symmetry plane (say y–z). By making use of the linear part of the displacement field (7′), the corresponding variations of the applied loads are obtained:

$$\mathbf{q}^{(1)} = (\mathbf{R}_q - \mathbf{I})\mathbf{q}, \quad \mathbf{f}^{(1)} = (\mathbf{R}_f - \mathbf{I})\mathbf{f}, \tag{19}$$

where $\mathbf{I}$ is the unit tensor and the first order rotation tensors $\mathbf{R}_q$ and $\mathbf{R}_f$ may assume either of the following forms:

$$\mathbf{R}_G = \begin{bmatrix} 1 & -\theta & \xi' - (y - y_S)\theta' \\ \theta & 1 & \eta' \\ -\xi' + (y - y_S)\theta' & -\eta' & 1 \end{bmatrix}, \tag{20}$$

$$\mathbf{R}_S = \begin{bmatrix} 1 & -\theta & \varphi_x \\ \theta & 1 & \varphi_y \\ -\varphi_x & -\varphi_y & 1 \end{bmatrix}. \tag{21}$$

The tensor $\mathbf{R}_G$ derives from the assumption that, during the deformation, the load follows the y–z plane keeping the same angle with the transformed local generatrix; on the other hand, $\mathbf{R}_S$ reflects the assumption that the load is rigidly connected with the middle plane of the cross-section. In the following, the cases $\mathbf{R}_q \equiv \mathbf{R}_G$ and $\mathbf{R}_f \equiv \mathbf{R}_S$ will be developed in detail whereas a numerical application will be given for the case $\mathbf{R}_f \equiv \mathbf{R}_G$.

With reference to a thin-walled beam element, the non-vanishing components of the Cauchy stress tensor $\boldsymbol{\sigma}$, corresponding to the static equilibrium configuration and referred to the local reference system, may be written as follows:

$$\sigma_{zs} = -\frac{T_y S_x}{J_x t} - \frac{T_{\psi_y} S_{\psi_y}}{J_{\psi_y} t}, \quad \sigma_{zz} = \frac{N}{A} - \frac{M_x}{J_x} y + \frac{M_{\psi_y}}{J_{\psi_y}} \psi_y \tag{22}$$

where

$$M_{\psi_y} = \int_A E\varepsilon_{zz}\psi_y \, \mathrm{d}A, \quad S_{\psi_y} = \int_0^s \psi_y t \, \mathrm{d}s, \quad J_{\psi_y} = \int_A (\psi_y)^2 \, \mathrm{d}A \tag{23}$$

and T_{ψ_y} is the first derivative of M_{ψ_y} with respect to z; finally the positive components of the beam forces are shown in Fig. 1.

5. STABILITY CRITERION

The basic method of investigating the stability of non-conservative problems consists of the analysis of the oscillations of the system close to its equilibrium position. An integral formulation governing the motion of a body B can be expressed through Hamilton's law† (Bailey, 1975):

† A superimposed dot means material derivative with respect to time.

$$\int_{t_1}^{t_2} (\delta T - \delta W + \delta L)\,\mathrm{d}t + \left[\int_B \rho \dot{\mathbf{u}}^{(1)} \cdot \delta \mathbf{u}^{(1)}\,\mathrm{d}V \right]_{t_1}^{t_2} = 0, \tag{24}$$

for any admissible variation of the generalized displacement functions. In eqn (24) δT is the variation of the kinetic energy and δW is the variation of the second order part of the strain energy. Moreover, δL represents a second order work collecting the work done by the conservative loads for the displacement components $\mathbf{u}^{(2)}$ and the further work done by the non-conservative components $\mathbf{f}^{(1)}$ and $\mathbf{q}^{(1)}$ of the external loads for the displacement field $\mathbf{u}^{(1)}$. The last term of eqn (24) containing the virtual work of impulses may be made equal to zero by imposing the requirement that $\delta\mathbf{u}^{(1)}(t_1) = \delta\mathbf{u}^{(1)}(t_2) = 0$ which is fully compatible with the assumption of free oscillations of the system. The variational equation (24) represents a non-self-adjoint problem since the external work δL cannot, generally, be derived from a potential. The residual terms appearing in eqn (24) may be given the compact expressions:

$$\delta T = \int_B \rho \dot{\mathbf{u}}^{(1)} \cdot \delta \dot{\mathbf{u}}^{(1)}\,\mathrm{d}V$$

$$\delta W = \int_B \boldsymbol{\varepsilon}^{(1)} \cdot \mathbf{E}\delta\boldsymbol{\varepsilon}^{(1)}\,\mathrm{d}V + \int_B \mathbf{S} \cdot \delta\boldsymbol{\varepsilon}^{(2)}\,\mathrm{d}V$$

$$\delta L = \int_{\partial B} \mathbf{p} \cdot \delta\mathbf{u}^{(2)}\,\mathrm{d}S + \int_{\partial B} \mathbf{p}^{(1)} \cdot \delta\mathbf{u}^{(1)}\,\mathrm{d}S, \tag{25}$$

where $\mathbf{E}$ is the elastic tensor at the origin and $\mathbf{S}$ is the second Piola–Kirchhoff stress tensor which, under the assumption of small components of both displacements and displacement gradients, reduces to the Cauchy stress tensor $\boldsymbol{\sigma}$. Moreover $\delta\mathbf{u}^{(2)}$ is the quadratic part of the displacement field (7″), $\mathbf{p}$ represents the loads acting on the body surface in the static equilibrium configuration and $\mathbf{p}^{(1)}$ expresses the variation of the external loads corresponding to the linearized displacement field $\mathbf{u}^{(1)}$.

Making use of eqns (7″), (10), (17), (18), (19) and (22) and keeping in mind that the load $\mathbf{p}$ in eqns (25) symbolizes the loads $\mathbf{q}$ and $\mathbf{f}$, the eqns (25) may be rewritten as follows:†

$$\delta T = \int_0^l \rho\{A\dot{\xi}\delta\dot{\xi} + A\dot{\eta}\delta\dot{\eta} + A\dot{\zeta}\delta\dot{\zeta} + y_S A\delta(\dot{\xi}\dot{\theta}) + J_S\dot{\theta}\delta\dot{\theta} + J_x\dot{\varphi}_x\delta\dot{\varphi}_x$$
$$+ J_y\dot{\varphi}_y\delta\dot{\varphi}_y + J_\omega\dot{\varphi}_\omega\delta\dot{\varphi}_\omega + J_{\psi_x}\dot{\chi}_x\delta\dot{\chi}_x + J_{\psi_y}\dot{\chi}_y\delta\dot{\chi}_y + J_{\psi_\omega}\dot{\chi}_\omega\delta\dot{\chi}_\omega\}\,\mathrm{d}z \tag{26}$$

$$\delta W = \int_0^l \{E[A\zeta'\delta\zeta' + J_x\varphi_x'\delta\varphi_x' + J_y\varphi_y'\delta\varphi_y' + J_\omega\varphi_\omega'\delta\varphi_\omega' + J_{\psi_x}\chi_x'\delta\chi_x'$$
$$+ J_{\psi_y}\chi_y'\delta\chi_y' + J_{\psi_\omega}\chi_\omega'\delta\chi_\omega'] + G[J_t\theta'\delta\theta' + D_x(\xi' - \varphi_y)\delta(\xi' - \varphi_y)$$
$$+ D_y(\eta' - \varphi_x)\delta(\eta' - \varphi_x) + D_\omega(\theta' - \varphi_\omega)\delta(\theta' - \varphi_\omega)$$
$$+ D_{\psi_x}\chi_x\delta\chi_x + D_{\psi_y}\chi_y\delta\chi_y + D_{\psi_\omega}\chi_\omega\delta\chi_\omega + D_{x\omega}\delta((\xi' - \varphi_y)(\theta' - \varphi_\omega))$$
$$+ D_{x\psi_x}\delta(\chi_x(\xi' - \varphi_y)) + D_{\omega\psi_x}\delta(\chi_x(\theta' - \varphi_\omega)) + D_{\psi_x\psi_\omega}\delta(\chi_x\chi_\omega)$$
$$+ D_{y\psi_y}\delta(\chi_y(\eta' - \varphi_x)) + D_{x\psi_\omega}\delta(\chi_\omega(\xi' - \varphi_y)) + D_{\omega\psi_\omega}\delta(\chi_\omega(\theta' - \varphi_\omega))]\}\,\mathrm{d}z$$
$$+ \int_0^l \{N[\xi'\delta\xi' + \eta'\delta\eta' + y_S\delta(\xi'\theta') + (J_S/A)\theta'\delta\theta']$$
$$- T_y\delta(\xi'\theta) + M_x\delta(\xi'\theta') + 2(C_x M_x + C_{\psi_x} M_{\psi_x})\theta'\delta\theta' - (C_x T_y + C_{\psi_x} T_{\psi_x})\delta(\theta\theta')\}\,\mathrm{d}z \tag{27}$$

† In the expression of δW, as usual in the beam theory, the Poisson effect has been accounted for by assuming $\varepsilon_{ss} = \varepsilon_{nn} = -\nu\varepsilon_{zz}$ whereas in eqn (18″), ε_{ss} and ε_{nn} were assumed to vanish.

$$\delta L = \int_0^l \{q_y[(-\theta\delta\xi - \eta'\delta\zeta) + (\eta' - \varphi_x)y\delta\varphi_x] + q_z[\eta'\delta\eta + (\xi' - (y - y_S)\theta')(\delta\xi - (y - y_S)\delta\theta) + (y - y_S)\delta(\theta\varphi_x)]\}\,\mathrm{d}z + [F_y(-\theta\delta\xi - \varphi_x\delta\zeta) + F_z(\varphi_x\delta\xi + \varphi_x\delta\eta - y_S\theta\delta\varphi_x) + \mathscr{M}_x\theta\delta\varphi_x]_{z=0,l}, \quad (28)$$

where E and G are the normal and tangential elastic moduli, q_y and q_z are the line load components in the y and z directions, F_y, F_z and $\mathscr{M}_x$ are the beam resultants of the external forces **f**, J_t is the torsion constant, J_S is the polar moment of inertia about the shear center and where:†

$$J_{ij} = \int_A ij\,\mathrm{d}A \quad (i, j = \psi_x, \psi_y, \psi_\omega) \quad (29)$$

$$D_{ij} = \int_A \frac{\mathrm{d}i}{\mathrm{d}s}\frac{\mathrm{d}j}{\mathrm{d}s}\,\mathrm{d}A \quad (i, j = x, y, \omega, \psi_x, \psi_y, \psi_\omega) \quad (30)$$

$$C_x = y_S - \frac{1}{2J_x}\int_A y(x^2 + y^2)\,\mathrm{d}A \quad (31)$$

$$C_{\psi_x} = -\frac{1}{2J_{\psi_x}}\int_A \psi_x(x^2 + y^2)\,\mathrm{d}A. \quad (32)$$

Equation (28) has been derived neglecting all the warping terms of the displacement field consistently with the assumption that the follower forces are insensitive to the warping terms. All the unknown functions have to belong to $H^1(0, l)$ according to the requirement of the variational formulation (24)–(28).

To summarize, the present formulation neglects flexural extensional and torsional extensional internal energy terms as well as the flexural–flexural terms given by the geometric effects of an applied torque. These contributions were evaluated in Laudiero and Zaccaria (1988a,b).

6. THE EQUATIONS OF MOTION

With reference to Hamilton's law expressed by eqn (24), making use of eqns (26)–(28) and integrating by parts yield the following equations of motion:

$$\rho A\ddot{\xi} + \rho y_S A\ddot{\theta} - GD_x(\xi' - \varphi_y)' - GD_{x\omega}(\theta' - \varphi_\omega)' - GD_{x\psi_x}\chi_x' - GD_{x\psi_\omega}\chi_\omega' - N\xi'' - Ny_S\theta'' + T_x\theta' - M_x\theta'' + q_y\theta - q_z\xi' + q_z(y - y_S)\theta' = 0$$

$$\rho A\ddot{\eta} - GD_y(\eta' - \varphi_x)' - GD_{y\psi_y}\chi_y' - N\eta'' - q_z\eta' = 0$$

$$\rho A\ddot{\zeta} - EA\zeta'' + q_y\eta' = 0$$

$$\rho J_S\ddot{\theta} + \rho y_S A\ddot{\xi} - GJ_t\theta'' - GD_\omega(\theta' - \varphi_\omega)' - GD_{x\omega}(\xi' - \varphi_y)' - GD_{\omega\psi_x}\chi_x' - GD_{\omega\psi_\omega}\chi_\omega' - Ny_S\xi'' - N(J_S/A)\theta'' - T_x\xi' - M_x\xi'' - 2(C_xM_x + C_{\psi_x}M_{\psi_x})\theta'' - q_z(y - y_S)^2\theta' + q_z(y - y_S)(\xi' - \varphi_y) = 0$$

$$\rho J_x\ddot{\varphi}_x - EJ_x\varphi_x'' - GD_y(\eta' - \varphi_x) - GD_{y\psi_y}\chi_y - q_y y(\eta' - \varphi_x) = 0$$

† The coupling term $J_{\psi_x\psi_y}$ has been disregarded in eqns (27) and (28) since it is negligible with respect to analogous coupling terms. Repeated subscripts are reported once only.

$$\rho J_y\ddot{\varphi}_y - EJ_y\varphi''_y - GD_x(\xi'-\varphi_y) - GD_{x\omega}(\theta'-\varphi_\omega) - GD_{x\psi_x}\chi_x - GD_{x\psi_\omega}\chi_\omega - q_z(y-y_S)\theta = 0$$

$$\rho J_\omega\ddot{\varphi}_\omega - EJ_\omega\varphi''_\omega - GD_\omega(\theta'-\varphi_\omega) - GD_{x\omega}(\xi'-\varphi_y) - GD_{\omega\psi_x}\chi_x - GD_{\omega\psi_\omega}\chi_\omega = 0$$

$$\rho J_{\psi_x}\ddot{\chi}_x - EJ_{\psi_x}\chi''_x + GD_{\psi_x}\chi_x + GD_{x\psi_x}(\xi'-\varphi_y) + GD_{\omega\psi_x}(\theta'-\varphi_\omega) + GD_{\psi_x\psi_\omega}\chi_\omega = 0$$

$$\rho J_{\psi_y}\ddot{\chi}_y - EJ_{\psi_y}\chi''_y + GD_{\psi_y}\chi_y + GD_{y\psi_y}(\eta'-\varphi_x) = 0$$

$$\rho J_{\psi_\omega}\ddot{\chi}_\omega - EJ_{\psi_\omega}\chi''_\omega + GD_{\psi_\omega}\chi_\omega + GD_{\psi_x\psi_\omega}\chi_x + GD_{x\psi_\omega}(\xi'-\varphi_y) + GD_{\omega\psi_\omega}(\theta'-\varphi_\omega) = 0, \quad (33)$$

with the relevant boundary conditions:

$$[(GD_x(\xi'-\varphi_y) + GD_{x\omega}(\theta'-\varphi_\omega) + GD_{x\psi_x}\chi_x + GD_{x\psi_\omega}\chi_\omega + N\xi'$$
$$+ Ny_S\theta' - T_y\theta + M_x\theta' + F_y\theta - F_z\varphi_y)\delta\xi]_0^l = 0$$

$$[(GD_y(\eta'-\varphi_x) + GD_{y\psi_y}\chi_y + N\eta' - F_z\varphi_x)\delta\eta]_0^l = 0$$

$$[(EA\zeta' + F_y\varphi_x)\delta\zeta]_0^l = 0$$

$$[(GJ_t\theta' + GD_\omega(\theta'-\varphi_\omega) + GD_{x\omega}(\xi'-\varphi_y) + GD_{\omega\psi_x}\chi_x + GD_{\omega\psi_\omega}\chi_\omega$$
$$+ Ny_S\xi' + N(J_S/A)\theta' + M_x\xi' + 2(C_xM_x + C_{\psi_y}M_{\psi_y})\theta' - (C_xT_y + C_{\psi_y}T_{\psi_y})\theta)\delta\theta]_0^l = 0$$

$$[(EJ_x\varphi'_x)\delta\varphi_x]_0^l = 0$$

$$[(EJ_y\varphi'_y - F_z(y-y_S)\theta)\delta\varphi_y]_0^l = 0$$

$$[(EJ_\omega\varphi'_\omega)\delta\varphi_\omega]_0^l = 0$$

$$[(EJ_{\psi_x}\chi'_x)\delta\chi_x]_0^l = 0$$

$$[(EJ_{\psi_y}\chi'_y)\delta\chi_y]_0^l = 0$$

$$[(EJ_{\psi_\omega}\chi'_\omega)\delta\chi_\omega]_0^l = 0. \quad (34)$$

7. THE DISCRETE MODEL

The solution to the problem (24) is obtained by making use of trigonometric series expansions so as to adhere strictly to the problem of the beam vibrations. Such a choice makes the comparison with the analogous results given by the Timoshenko and Euler–Bernoulli models more direct.

All the boundary conditions which are not satisfied by the coordinate functions will be enforced by a projection onto a proper subspace.

Assuming the displacements to be functions of time and space independently, each of the unknown functions and the corresponding variation can be given the compact form:

$$g(z,t) = \mathbf{a}_g^T\bar{\mathbf{g}}(z)\,e^{ipt}$$
$$\delta g(z,t) = \delta\mathbf{a}_g^T\bar{\mathbf{g}}(z)\,e^{ipt} \quad (35)$$

where the vectors $\bar{\mathbf{g}}$ and $\mathbf{a}_g$ contain a suitable number of coordinate functions and the corresponding Lagrangian coordinates respectively.

Substituting eqns (35) into (26)–(28), the variations of the kinetic and strain energy and that of the work of the external loads can be written in the form:

$$\delta T = p^2\delta\mathbf{a}^T\mathbf{M}\mathbf{a}\,e^{2ipt}$$
$$\delta W = \delta\mathbf{a}^T\mathbf{K}_w\mathbf{a}\,e^{2ipt}$$
$$\delta L = -\delta\mathbf{a}^T\mathbf{K}_L\mathbf{a}\,e^{2ipt} \quad (36)$$

where the vector $\mathbf{a}$ represents a global vector of unknown generalized coordinates collecting all the vectors $\mathbf{a}_{ij}$, the matrix $\mathbf{K}_L$ is non-symmetric since δL contains the work of the follower loads and the matrix $\mathbf{M}$ is the mass matrix.

Hence, setting $\mathbf{K} = K_W + \mathbf{K}_L$, the variational equation (24) turns out to be discretized as follows:

$$\int_{t_1}^{t_2} \delta\mathbf{a}^T(\mathbf{K}\mathbf{a} - p^2\mathbf{M}\mathbf{a})\, e^{2ipt}\, dt = 0 \tag{37}$$

subjected to:

$$\mathbf{C}\mathbf{a}\, e^{2ipt} = 0, \quad \mathbf{C}\delta\mathbf{a}\, e^{2ipt} = 0 \tag{38}$$

where eqn (38) collects the boundary conditions which are not satisfied *a priori* by the functions $\bar{\mathbf{g}}$. By decomposing the matrix $\mathbf{M}$ according to a Cholesky scheme, i.e. $\mathbf{M} = \mathbf{L}\mathbf{L}^T$ and setting $\mathbf{x} = \mathbf{L}^T\mathbf{a}$, the problem (37), (38) becomes:

$$\int_{t_1}^{t_2} \delta\mathbf{x}^T(\mathbf{A}\mathbf{x} - p^2\mathbf{x})\, e^{2ipt}\, dt = 0 \tag{39}$$

subjected to:

$$\mathbf{B}\mathbf{x} = 0, \quad \mathbf{B}\delta\mathbf{x} = 0 \tag{40a,b}$$

where

$$\mathbf{A} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}, \quad \mathbf{B} = \mathbf{C}\mathbf{L}^{-T}. \tag{41}$$

In order to satisfy the constraints (40), a projection operator $\mathscr{P}$ is defined (Aoki, 1971):

$$\mathscr{P} = \mathscr{P}^T = \mathbf{I} - \mathbf{B}(\mathbf{B}\mathbf{B}^T)^{-1}\mathbf{B}, \tag{42}$$

which projects any given vector $\mathbf{x} \in \mathbb{R}^N$ onto the subspace defined by the linear homogeneous equation (40a). Hence, making use of the condition $\mathscr{P} = \mathscr{P}^2$, the problem (39), (40) is reduced to the form:

$$\int_{t_1}^{t_2} \delta\mathbf{x}^T\mathscr{P}(\mathscr{P}\mathbf{A}\mathscr{P}\mathbf{x} - p^2\mathscr{P}\mathbf{x})\, e^{2ipt}\, dt = 0. \tag{43}$$

Since in the quotient space $\mathbb{R}^N/\mathscr{P}$ the variation $\delta\mathscr{P}\mathbf{x}$ is arbitrary, from eqn (43) it is immediately obtained:

$$\mathscr{P}\mathbf{A}\mathscr{P}\mathbf{x} - p^2\mathscr{P}\mathbf{x} = 0. \tag{44}$$

Equation (44) will be rewritten in the following:

$$\mathscr{A}\mathbf{y} - \lambda\mathbf{y} = 0, \tag{45}$$

where $\lambda = p^2$, $\mathscr{A} = \mathscr{P}\mathbf{A}\mathscr{P}$ and $\mathbf{y} = \mathscr{P}x \in \mathbb{R}^N/\mathscr{P}$, as required by the original problem.

Equation (45) represents a standard eigenvalue problem which satisfies all the boundary conditions, including the non-essential ones. This technique turns out to be particularly efficient since the structural response will be extremely sensitive to the loading features as will be seen in the following.

Since the matrix **K**, and consequently the matrix $\mathcal{A}$, are strongly ill-conditioned and non-symmetric, it was necessary to develop a suitable algorithm (Alliney *et al.*, 1990) which is briefly outlined hereafter. The mechanical counterpart of the ill-conditioning of the matrix **K** involves the rapid increase of the circular frequencies (square roots of the corresponding eigenvalues) with the higher modes of vibration.

We will restrict ourselves to the search of the real solutions to the problem (45); hence the analysis will be developed under the assumption that the eigenvalue problem admits one or more real solutions. In the framework of the least squares method, the solutions will be derived from the condition:

$$\min J(\mathbf{y},\lambda) \mid J(\mathbf{y},\lambda) = \frac{\|\mathcal{A}\mathbf{y}-\lambda\mathbf{y}\|_2^2}{\|\mathbf{y}\|_2^2} \tag{46}$$

where $J \geqslant 0$ vanishes iff the pair $(\mathbf{y},\lambda)$ is a solution of the problem (45). It can be shown that the problem (46) can be given the equivalent, Rayleigh-type, form:

$$\min \mathcal{R}(\mathbf{y}) \mid \mathcal{R}(\mathbf{y}) = \frac{\mathbf{y}^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}\mathbf{Q}(\mathbf{y})\mathcal{A}\mathbf{y}}{\|\mathbf{y}\|_2^2} \tag{47}$$

where **Q**, the Householder reflector (Stewart, 1973), is given by the expression:†

$$\mathbf{Q}(\mathbf{z}) = \mathbf{I}-\hat{\mathbf{z}}\hat{\mathbf{z}}. \tag{48}$$

The minimization of $\mathcal{R}(\mathbf{y})$ is obtained by means of a vector iteration technique. At the nth iteration the Householder reflector is determined by means of the nth trial vector $\mathbf{y}_n$ giving $\mathbf{Q}_n = \mathbf{Q}(\mathbf{y}_n)$; then the gradient of the form:

$$\mathcal{R}_n(\mathbf{y}) = \frac{\mathbf{y}^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}\mathbf{Q}_n\mathcal{A}\mathbf{y}}{\|\mathbf{y}\|_2^2} \tag{49}$$

is determined, at the point $\mathbf{y}_n$, resulting (to the accuracy of a scalar multiplier) in the expression:

$$\mathbf{r}_n = \mathcal{A}^{\mathrm{T}}\mathbf{Q}_n\mathcal{A}\hat{\mathbf{y}}_n-(\hat{\mathbf{y}}_n^{\mathrm{T}}\mathcal{A}^{\mathrm{T}}\mathbf{Q}_n\mathcal{A}\hat{\mathbf{y}})\hat{\mathbf{y}}_n. \tag{50}$$

Hence, a step is performed towards the minimization of $R(\mathbf{y})$ by correcting the current vector as follows:

$$\mathbf{y}_{n+1} = \hat{\mathbf{y}}_n-\alpha\hat{\mathbf{r}}_n \tag{51}$$

where α may be determined through the condition that $\mathcal{R}_n(\hat{\mathbf{y}}_n)-\mathcal{R}_n(\hat{\mathbf{y}}_{n+1})$ is maximum, resulting in:

$$\alpha_n = \tfrac{1}{2}\{[\mathcal{R}_n(\hat{\mathbf{y}}_n)-\mathcal{R}_n(\hat{\mathbf{r}}_n)]+\sqrt{[\mathcal{R}_n(\hat{\mathbf{y}}_n)-\mathcal{R}_n(\hat{\mathbf{r}}_n)]^2+4\|\mathbf{r}_n\|_2}\}. \tag{52}$$

† The symbol (^) indicates normalization of the vector acted upon.

8. NUMERICAL RESULTS

All the examples solved are constituted by cantilever beams having a cross-section corresponding to one of the two shown in Fig. 5, with the exception of the first example which does not require such a specification. The warping functions ψ_x, ψ_y and ψ_ω were obtained by making use of relations (12)–(15) and are reported in the Appendix together with the series expansions adopted for the displacement functions.

The first example refers to the classical Beck problem which was solved by adopting the Euler–Bernoulli model. The lateral displacement was interpolated with an increasing number of terms of the series expansion and the rate of convergence is shown in Table 1. The flutter loads were obtained by checking the trajectories of the first two circular frequencies while increasing the applied load. The starting trial vectors to be used at each load level were assumed to coincide with the optimal vectors of the preceding load level.

Subsequently, the same problem was solved by adopting 16 terms of the series expansion for each unknown function in order to compare the results of the proposed model with those given by the Euler–Bernoulli and Timoshenko models (Table 2). When the beam

Table 1. Convergence test (Beck problem)

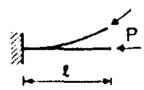
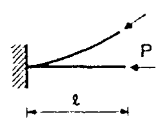
NUMBER OF TERMS	8	16	32	48	
$P_{cr}\,\ell^2/EJ$	20.0068	20.0419	20.0496	20.0505	EULER–BERNOULLI $P_{cr}\ell^2/EJ = 20.0509$

Table 2. Influence of slenderness on flexural flutter load (Beck problem)

ℓ/h	2.50	5.00	10.00	15.00	
EULER–BERNOULLI	20.04	20.04	20.04	20.04	
TIMOSHENKO	2.065	4.661	10.55	14.23	
PRESENT ANALYSIS	2.207	4.777	10.72	14.37	I Section b = h $\dfrac{P_{cr}\ell^2}{EJ_x}$

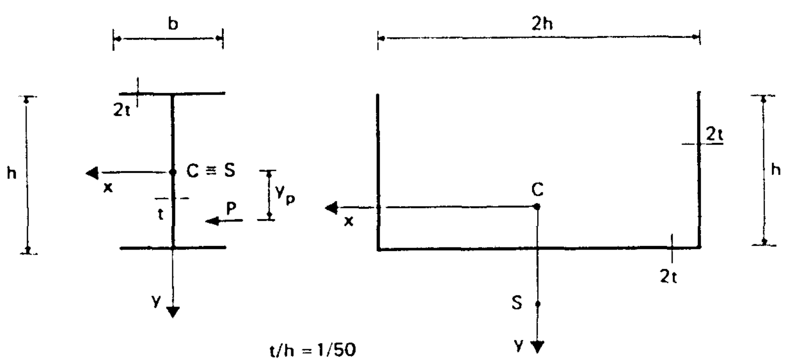


Fig. 5. Cross-sections considered for the numerical examples.

Table 3. Influence of slenderness on flexural flutter load (Leipholz problem).

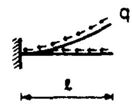
ℓ/h	2.50	5.00	10.00	15.00	
CLASSICAL MODEL	40.05	40.05	40.05	40.05	
PRESENT ANALYSIS	3.914	7.265	19.17	27.26	I Section b = h $\frac{q_{cr}\ell^3}{EJ_x}$

Table 4. Influence of slenderness on torsional flutter load (Nemat-Nasser and Herrmann problem).

ℓ/h	2.50	5.00	10.00	15.00	
CLASSICAL MODEL	18.56	18.86	20.70	22.20	
PRESENT ANALYSIS	10.44	15.42	19.09	21.75	I Section b = h $\frac{P_{cr}\ell^2 J_c}{EA J_w}$

becomes rather short, it can be seen that preventing the built-in section from warping makes the beam stiffer with respect to the Timoshenko beam model.

The third example refers to a cantilever beam subjected to a uniformly distributed axial load which follows the deformed centroidal axis (Leipholz problem). The numerical results reported in Table 3 confirm the determining role that the shear strains play, especially in the higher modes of the flexural vibrations (16 terms were adopted for each unknown function).

The fourth example analyzes the influence of the shear strains on the torsional flutter load for a cantilever beam (having an I cross-section) subjected to two axial forces acting at the intersections between the web and the flanges of the end cross-section (Nemat-Nasser and Herrmann problem). The two forces are assumed to follow the local transformed generatrices. The numerical results reported in Table 4 (16 terms for each unknown function) show that the shear strains still play a significant role even if not such a dramatic one as in the case of the flexural flutter.

In the fifth example, a cantilever beam having an I cross-section is subjected to an end axial force with three different eccentricities. The case $y_p = 0$ corresponds to a force acting at the centroid of the cross-section and, consequently, the first column of Table 5 refers to uncoupled modes of vibration. The case $y_p = 0.5h$ corresponds to a force acting at the intersection of the web with a flange. Hence, the second and third columns refer to coupled modes of vibrations and the symbols T (torsional) and F (flexural) denote the dominant component of the vibration mode. The numerical results were obtained by adopting eight terms for each unknown function. The cases corresponding to the first three rows of Table 5 were analyzed by Nemat-Nasser and Tsai (1969).

In the subsequent examples the trajectories themselves of the non-dimensional circular frequencies are shown in order to emphasize a phenomenon already revealed by the previous example, i.e. how the critical response may be characterized by buckling or flutter depending whether the load follows the middle plane of the cross-section or the local generatrix. The structural element considered was a cantilever beam with U section subjected to an end axial force; therefore, the flexural-torsional coupling was an inherent consequence of the structural geometry. The analogous problem (with no shear strain influence) has been recently studied by Aida (1986). Figure 6 shows how the critical response additionally depends on the slenderness ratio; in fact, it tends to the buckling mode for both the loading features as the beam length is reduced. In this example eight terms for each unknown function were adopted and the shear strain influence was fully considered.

Table 5. Critical values for a cantilever beam with an I section subjected to an eccentric axial follower force.

$\frac{P_{cr}\ell^2}{EJ_y}$	$y_p = 0.$ Mode		$y_p = 0.25$ h Mode		$y_p = 0.50$ h Mode		ℓ/h = 5.0 b = h
The load follows the local transformed generatrix	T–Div	2.407	T–Div	3.330	F-Flu	9.516	Classical model
	T-Div	19.82	F-Flu	10.61	T-Div	38.73	
	F-Flu	20.01					
	T-Div	2.353	T-Div	3.269	F-Flu	7.285	Present Analysis
	F-Flu	15.15	F-Flu	8.912	T-Div	39.18	
	T-Div	16.16					
The load remains orthogonal to the middle plane of the cross-section	T-Div	2.407	T-Div	2.548	T-Div	3.438	Classical model
	T-Div	19.82	F-Flu	10.02	F-Flu	7.566	
	F-Flu	20.01					
	T-Div	2.353	T-Div	2.484	T-Div	3.339	Present Analysis
	F-Flu	15.12	F-Flu	8.253	F-Flu	6.357	
	T-Div	16.16					

T (torsional) and F (flexural) indicate the vibration mode (possibly the dominant component). Div stands for divergence and Flu for flutter.

Finally, the case of a cantilever beam, with an I cross-section, subjected to an end lateral force was considered. Again (out-of-plane) flexural and torsional modes turn out to be coupled. For the sake of simplicity, the shear strain influence was not considered in this example, for which eight terms for each unknown function were retained. This problem does not seem to have been solved in the literature since only thin rectangular cross-sections were considered (Como, 1966; Barsoum, 1971; Attard and Somervaille, 1987). In Fig. 7 the non-dimensional values of the flutter load were plotted against the ratio of the first two (flexural and torsional) frequencies of the unloaded structure. The relative variation of the two frequencies was obtained by assuming different ratios of the flange width over the web height. Figure 7 shows how the flutter load approaches zero as the frequency ratio tends to unity. This paradoxical result was already underlined by Bolotin (1963) who noticed that, when the natural frequencies of the unloaded structure approach each other, even slight damping may have a stabilizing effect. In these cases, however, a non-linear analysis which goes beyond the limits of a linearized formulation (Sethna and Shapiro, 1977) seems to be the required development.

9. CONCLUSIONS

A model has been presented for thin-walled beams of open section which takes into account the middle surface shear deformations associated with shear forces as well as with secondary torsion. All the loads have been assumed to act in a symmetry plane of the beam according to two distinct features, i.e. the loads may follow either the middle plane of the cross-section or the local deformed generatrix. The internal stresses have been directly derived from the kinematic field by making use of the constitutive relations. Following the usual separation of space and time variables, the variational equation expressing Hamilton's law has been discretized by making use of trigonometric series expansions. This choice

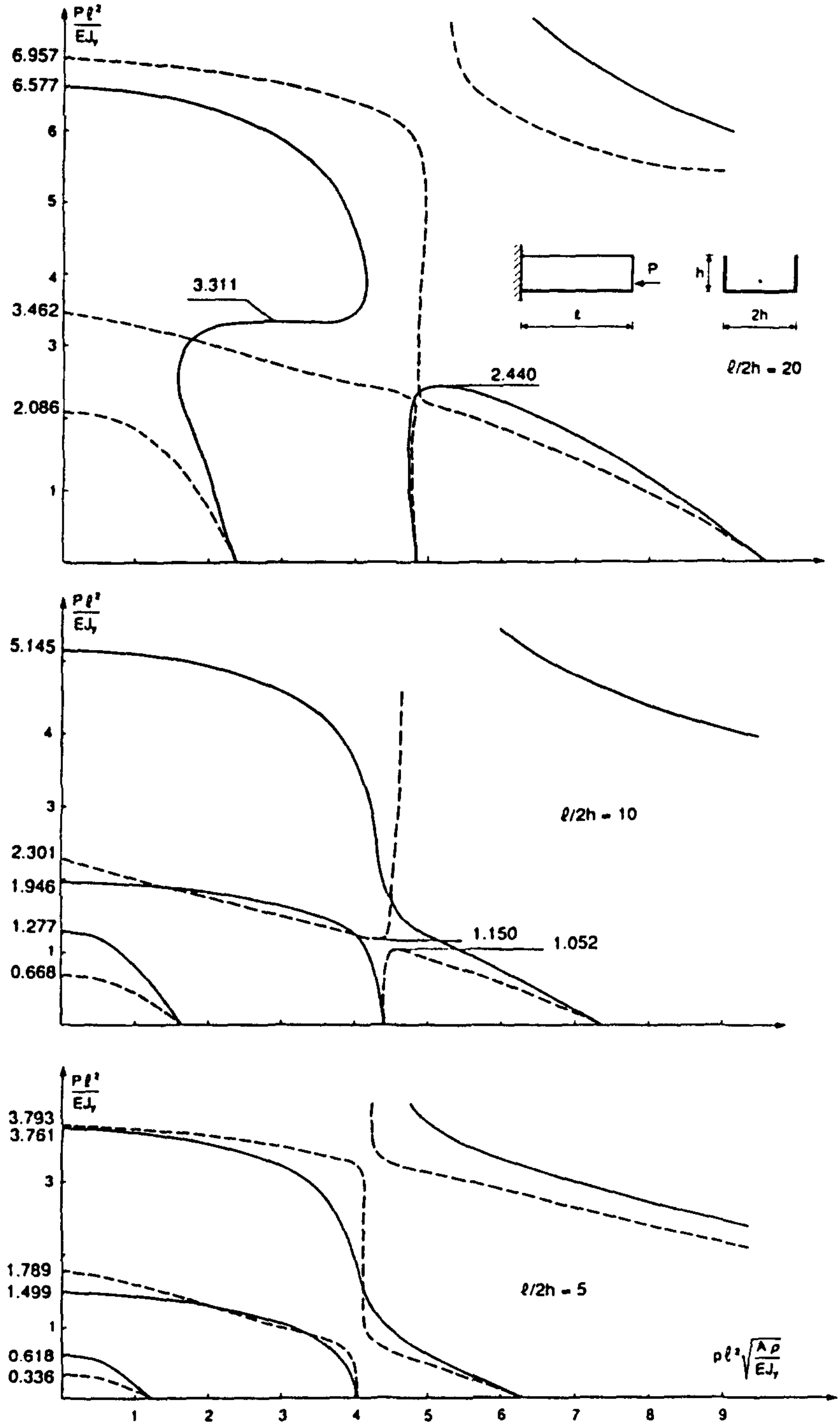


Fig. 6. Circular frequencies of flexural torsional vibration modes for a cantilever beam subjected to an axial force. The dashed line indicates that the force remains orthogonal to the cross-section. The solid line indicates that the force follows the local transformed generatrix.

appears to be the most strictly related to the features of a vibrating beam. For the numerical applications a suitable algorithm has been developed, based on a vector iteration technique, which overcomes the difficulties arising from an eigenvalue problem ruled by a non-symmetric and strongly ill-conditioned matrix. A variety of examples clarify the significant influence of the shear deformations on the critical loads involving flexural as well as torsional modes.

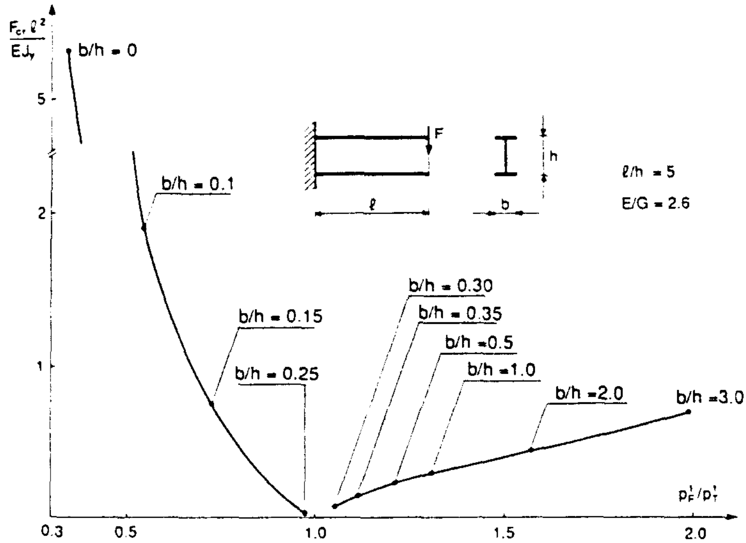


Fig. 7. Flexural–torsional flutter load versus the ratio between the first flexural and torsional circular frequencies of the unloaded structure.

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APPENDIX

With reference to the cross-sections adopted in the numerical examples, Figs A1 and A2 show the (normalized) warping functions ψ_x, ψ_y and ψ_ω determined for the I section and ψ_y and ψ_ω determined for the U section.

Finally, in the numerical examples the unknown functions were represented as follows:

$$\xi(z), \eta(z), \theta(z) \Rightarrow \sum_{n=1}^{N} a_n \left(\cos\left(\frac{2n-1}{2} \pi \frac{z}{l} \right) - 1 \right),$$

$$\varphi_x(z), \varphi_y(z), \varphi_\omega(z) \Rightarrow \sum_{n=1}^{N} a_n \sin\left(\frac{2n-1}{2} \pi \frac{z}{l} \right),$$

$$\chi_x(z), \chi_y(z), \chi_\omega(z) \Rightarrow \sum_{n=1}^{N} a_n \sin\left(\frac{2n-1}{2} \pi \frac{z}{l} \right).$$

These representations are able to satisfy all the essential boundary conditions of the examples considered.

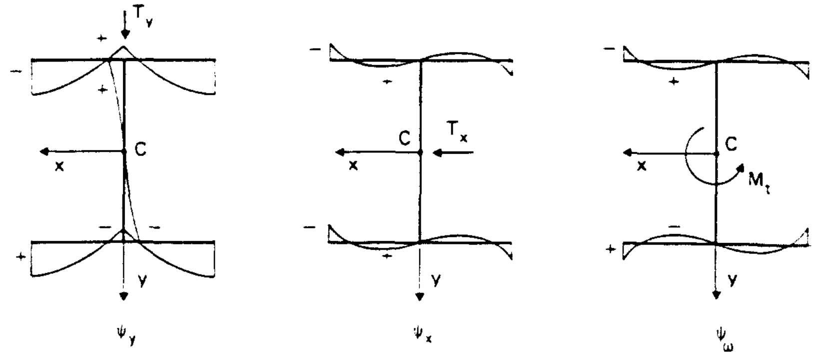

Fig. A1. Normalized warping functions ψ_y, ψ_x and ψ_ω for the I section.

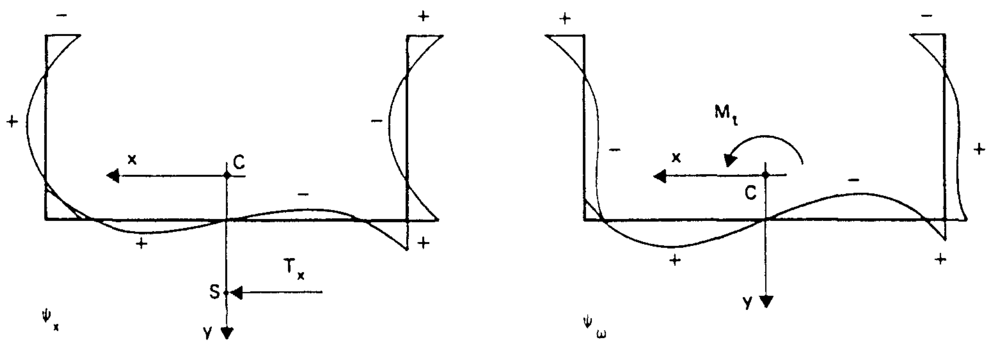

Fig. A2. Normalized warping functions ψ_x and ψ_ω for the U section.